Exact Wirelength of Hypercube and Enhanced Hypercube Layout on Wounded Lobsters

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ABSTRACT

Embeddings on various architectures are used not only to study the simulation capabilities of a parallel architecture but also to design its VLSI layout. In addition to dilation and congestion, wirelength is an important measure of an embedding. As far as the most versatile architecture hypercube is concerned, only approximate estimates of the wirelength of various embeddings are available. In this paper, we give an exact formula of minimum wirelength of hypercube and enhanced hypercube layout into wounded lobsters and thereby we solve completely the wirelength problem of hypercube and enhanced hypercube layout into wounded lobsters.

Key Words: Fixed interconnection parallel architecture, hypercube, enhanced hypercube, lobsters, embedding, wirelength.

1. Introduction and Terminology

A parallel algorithm or a massively parallel computer can be each modeled by a graph, in which the vertices of the graph represent the processes or processing elements, and the edges represent the communications among processes or processors. Thus, the problem of efficiently executing a parallel algorithm *A* on a parallel computer *M* can be often reduced to the problem of mapping the graph *G*, representing *A*, on the graph *H*, representing *M*, so that the mapping satisfies some predefined constraints. This is called graph embedding [14], which is defined more precisely as follows:

Let *G* and *H* be finite graphs with *n* vertices. *V*(*G*) and *V*(*H*) denote the vertex sets of *G* and *H* respectively. *E*(*G*) and *E*(*H*) denote the edge sets of *G* and *H* respectively. A 1-1 mapping *f*: *V*(*G*) → *V*(*H*) is called an embedding of *G* into *H*. See Figure 1. A set of edges of *H* is said to be an *edge cut* of *H* if the removal of these edges results in a disconnection of *H*.

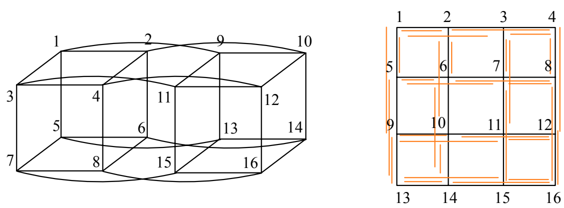

Figure 1: Embedding of a Hypercube onto a Grid

The *congestion* of an embedding *f* of *G* into *H* is the maximum number of edges of *G* that are embedded on any single edge of *H*. Let $EC_f(G,H(e))$ denote the number of edges (*u*, *v*) of *G* such that *e* is in the path $P_f(f(u),f(v))$ between *f*(*u*) and *f*(*v*) in *H*. In other words,

$$EC_f(G,H(e)) = \left|\{(u,v)\in E(G) / e \in P_f(f(u),f(v))\}\right|$$

where $P_f(f(u), f(v))$ denotes the path between $f(u)$ and $f(v)$ in H, with respect to f. Minimum of $EC_f(G, H(e))$, where the minimum is taken over all embeddings f is denoted by $C_{\min}(e)$.

1.1 The Edge Congestion Problem

The *edge congestion* [13, 17] of an embedding f of G into H is given by

$$EC_f(G,H) = max\ EC_f(G,H(e))$$

where the maximum is taken over all edges e of H. Then, the *minimum edge congestion* of G into H is defined as

$$EC(G,H) = min\ EC_f(G,H)$$

where the minimum is taken over all embeddings f of G into H. See Figure 2. The *edge congestion problem* [11, 15] of a graph G into H is to find an embedding of G into H that induces the minimum edge congestion $EC(G,H)$.

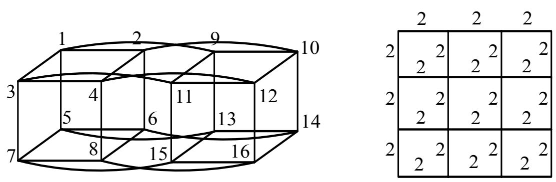

Figure 2: For the embedding mentioned in Figure 1, the edge congestions are marked on the respective edges of the Grid

1.2 The Wirelength Problem

The *wirelength* of an embedding f of G into H is given by

$$WL_f(G,H) = \sum EC_f(G,H(e))$$

where the sum is taken over all the edges e of H. Then, the *minimum wirelength* of G into H is defined as

$$WL(G,H) = min\ WL_f(G,H)$$

where the minimum is taken over all embeddings f of G into H. The *wirelength problem* [4, 5, 8, 11 14] of a graph G into H is to find an embedding of G into H that induces the minimum wirelength $WL(G,H)$. □

2. Overview of the Paper

The wirelength of a graph embedding arises from VLSI designs, data structures and data representations, networks for parallel computer systems, biological models that deal with cloning and visual stimuli, parallel architecture, structural engineering and so on [13, 17]. VLSI Layout Problem [1], is a part of grid embedding. Embedding problems have been considered for star networks into hypercubes [2], complete trees into hypercubes [3], hypercubes into grids [4, 16], generalized ladders into hypercubes [7], complete graphs into hypercubes [12], hypercubes into complete binary trees [15], and binary trees into grids [14].

Even though there are numerous results and discussions on the wirelength problem, most of them deal with only approximate results and the estimation of lower bounds [4, 8]. The embeddings discussed in this paper produce an exact wirelength. We derive a formula for the minimum wirelength of hypercube and enhanced hypercubes into wounded lobsters. □

3. A Few Basic Results

Here onwards, for the sake of simplicity $EC_f(G, H(e))$ will be represented by $EC_f(e)$. For any set S of edges of H, $EC_f(S) = \sum_{e \in S} EC_f(e)$. The following Lemma will be used throughout this paper to estimate the wirelength.

Lemma 1 (Congestion Lemma) [15]: Let f be an embedding of a regular graph G into an arbitrary tree T. Let $e \in E(T)$ and T_1 be a component of $T - e$. Then, the congestion $C_f(e)$ on e is given by

$$C_f(e) = \sum_{v \in G_1} d_G(v) - 2|E(G_1)|$$

where G_1 is a subgraph of G induced by the vertices $\{f^{-1}(u)/u \in T_1\}$ and $d_G(v)$ denotes the degree of v in G. More over $C_f(e)$ is minimum whenever G_1 is maximum. □

4. The Wirelength Problem of the Hypercubes Q^r into Wounded Lobsters

In this section we solve the wirelength problem of hypercubes into wounded lobsters.

Definition 1: For $r \geq 1$, let Q^r denote the graph of r-dimensional hypercube. The vertex set of Q^r is formed by the collection of all r-dimensional binary representations. Two vertices x, $y \in V(Q^r)$ are adjacent if and only if the corresponding binary representations differ exactly in one bit [3, 13, 17].

Remark: For our convenience, the labels {0, 1,…, $2^r - 1$} of Q^r are represented by {1, 2, …, 2^r} respectively.

Definition 2 [9]: A *caterpillar* is a tree with the property that the removal of its end points leaves a path called the *spine*. If r pendant edges are incident at every spine vertex then it is an *r-regular caterpillar*. A *lobster* is a tree with the property that the removal of end points leaves a caterpillar. A *wounded lobster* L^r is a lobster satisfying the following three conditions:

(i) There are 2^{r-2} spine vertices.
(ii) Every spine vertex is adjacent to exactly one vertex of degree 2.
(iii) Removal of pendant edges incident at vertices of degree 2 leaves a 2-regular caterpillar.

A wounded lobster L^r has 2^r vertices. See Figure 3.

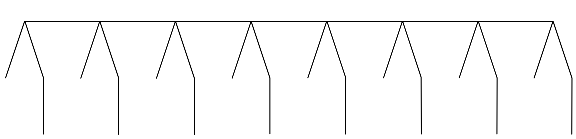

Figure 3: Wounded Lobster L^5

Embedding Algorithm *A*

The embedding f of Q^r with the lexicographic labeling 1 to 2^r into the wounded lobster L^r is an assignment of labeling of the vertices of the wounded lobster L^r as follows: For $j = 0,1,...,2^{r-2} - 1$

(i) Label the spine vertices as $4j+1$ from left to right.
(ii) Label the vertex of degree 1 adjacent to $4j+1$ as $4j+2$.
(iii) Label the vertex of degree 2 adjacent to $4j+1$ as $4j+3$.
(iv) Label the vertex adjacent to $4j+3$ as $4j+4$. See Figure 4.

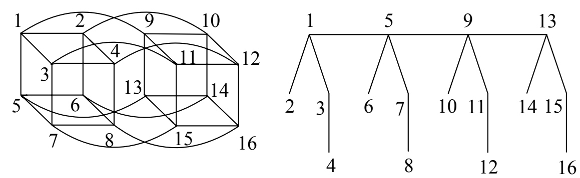

Figure 4: Embedding *f* of Q^4 into L^4

Theorem 1 [6, 9]: Let Q^r be an r-dimensional hypercube. For i = 1, 2, …, 2^r, the subgraph induced by $L_i = \{1,2,...,i\}$ has maximum number of edges on i vertices. □

Theorem 2: The embedding f defined by the Embedding Algorithm A yields minimum wirelength of Q^r into L^r.

Proof: We call an edge e of L^r as e_α if there exist a component L_α of $L^r - e$ on α vertices. The components L_2, which are 2^{r-2} in number are labeled $4j+3$ and $4j+4$, $j = 0,1,...,2^{r-2} - 1$. The inverse images of the vertices labeled $4j+3$ and $4j+4$, clearly induce a maximum subgraph of Q^r. This is true for every $j = 0,1,...,2^{r-2} - 1$. Similarly, each spine edge e_α has one component L_α labeled $1,2,\ldots,\alpha$. By Theorem 1, these labels induce a maximum subgraph in Q^r. Thus by Congestion Lemma the embedding induces minimum wirelength of Q^r into L^r. See Figure 5.

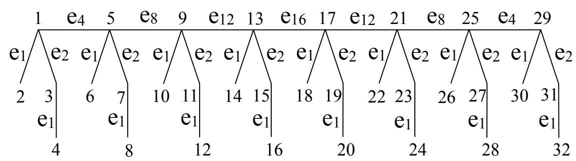

Figure 5: No of e_α 's in L^5

Theorem 3:
$WL(Q^r, L^r) = 2^{r-1}(2^{r-2} + 2r - 2)$.

Proof: By Theorem 2, we have $WL(Q^r, L^r) = WL_f(Q^r, L^r)$. Hence it is enough to compute $WL_f(Q^r, L^r)$. We prove the result by induction on *r*. Let *r* = 3. Maximum subgraph of Q^3 on 4 vertices has 4 edges. Hence by Lemma 1,

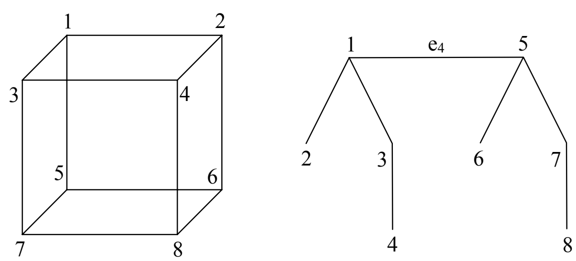

Figure 6: Embedding of Q^3 into L^3

$C_{\min}(e_4) = 3(4) - 2(4) = 4$, $C_{\min}(e_2) = 4$, $C_{\min}(e_1) = 3$, $\Psi(e_2) = 2$ and $\Psi(e_1) = 4$. See Figure 6. Hence, $WL_f(Q^3, L^3) = \Psi(e_1)$
$C_{\min}(e_1) + \Psi(e_2) C_{\min}(e_2) + \Psi(e_4)$
$C_{\min}(e_4)$
$= 4 \times 3 + 2 \times 4 + 1 \times 4 = 24$
$= 2^{3-1}(2^{3-2} + 2 \times 3 - 2)$.
Assume the result to be true for *r* = *k*. Consider Q^{k+1}. In L^{k+1}, there exist one edge e_{2^k} such that $L^{k+1} - e_{2^k}$ gives two wounded lobsters each isomorphic to L^k. By induction hypothesis $WL_f(Q^k, L^k) =$ $2^{k-1}(2^{k-2} + 2k - 2)$. Since the vertex labeled *i* is adjacent to vertex labeled $i + 2^k$ in Q^{k+1}, the edges of Q^{k+1} get dilated over edges of L^{k+1} as follows:

(i) There are 2^{k-2} edges $(4t-2, 4t-2+2^k)$, $1 \le t \le 2^{k-2}$ of Q^{k+1}, each dilated over $2^{k-2} + 2$ edges of L^{k+1} from a pendant vertex adjacent to a spine vertex to a corresponding pendant vertex.

(ii) There are 2^{k-2} edges $(4t-1, 4t-1+2^k)$, $1 \le t \le 2^{k-2}$ of Q^{k+1}, each dilated over $2^{k-2} + 2$ edges of L^{k+1} from a vertex of degree 2 adjacent to a spine vertex to a corresponding vertex of degree 2 in L^{k+1}.

(iii) There are 2^{k-2} edges $(4t, 4t+2^k)$, $1 \le t \le 2^{k-2}$ of Q^{k+1}, each dilated over $2^{k-2} + 4$ edges of L^{k+1} from a pendant vertex adjacent to vertex of degree 2 to a corresponding pendant vertex in L^{k+1}.

(iv) There are 2^{k-2} edges $(4t-3, 4t-3+2^k)$, $1 \le t \le 2^{k-2}$ of Q^{k+1}, each dilated over 2^{k-2} spine edges of L^{k+1}.

Hence,
$WL_f(Q^{k+1}, L^{k+1}) = 2WL_f(Q^k, L^k) + 2^{k-1}$
$(2^{k-2} + 2) + 2^{k-2}(2^{k-2} + 4) + 2^{k-2}2^{k-2}$
$= 2^{(k+1)-1}(2^{(k+1)-2} + 2(k+1) - 2)$.
Thus, $WL(Q^r, L^r) = 2^{r-1}(2^{r-2} + 2r - 2)$. □

5. Wirelength Problem of $Q_{r,k}$ into Wounded Lobsters

In this section we solve the wirelength problem of enhanced hypercubes into wounded lobsters.

Definition 3: The *enhanced hypercube* $Q_{r,k}$, $1 \le k \le r-1$ is an undirected graph with vertex set $V(Q_{r,k}) = V(Q^r)$ and edge set $E(Q_{r,k}) = E(Q^r) \cup \{(x_1x_2...x_{k-1}x_k x_{k+1}...x_r,\ x_1x_2...x_{k-1}\overline{x}_k\overline{x}_{k+1}...\overline{x}_r) : x_1x_2...x_r \in V(Q_{r,k})\}$.

Embedding Algorithm *B*

The embedding of $Q_{r,k}$, $1 \le k \le r-1$ with lexicographic labeling 1 to 2^r into wounded lobster L^r is the same as that of *f* described in Embedding Algorithm *A*. See Figure 7.

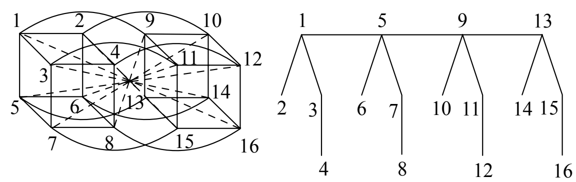

Figure 7: Embedding of *f* $Q_{4,1}$ into L^4

We have the following result analogous to Theorem 1.

Theorem 4 [6, 10]: Let $Q_{r,k}$ be an *r*-dimensional hypercube. For $i = 1,2,...,2^r$, $L_i = \{1,2,...,i\}$ has maximum number of edges on *i* vertices. □

The following result is analogous to Theorem 2.

Theorem 5: The embedding *f* defined by the Embedding Algorithm *B* yields minimum wirelength of $Q_{r,k}$ into L^r. □

The proof of the following Theorem is similar to that of Theorem 3.

Theorem 6:

$$WL(Q_{r,k}, L^r) = \begin{cases} 2^{r-3}(8r + 2^r + 2^{r-k}), & 1 \le k \le r-2 \\ 2^{r-3}(8r + 2^r), & k = r-1 \end{cases}$$

6. Conclusion

We solve the wirelength problem of hypercube and enhanced cube into wounded lobsters. It would be interesting to solve wirelength problem for architectures such as butterfly, torus, star, and pancake. □

References :

[1] S. N. Bhatt and F. T. Leighton, "A framework for solving VLSI graph layout problems", J. Computer and System Sciences, Vol. 28, 1984, pp. 300 - 343.

[2] S. Bettayeb, B. Cong, M. Girou and I. H. Sudborough, "Embedding of star networks into hypercubes", IEEE Transactions on Computers, Vol. 45, 1996, pp. 186 - 194.

[3] S. L. Bezrukov, "Embedding complete trees into the hypercube", Discrete Appl. Math., Vol. 110, 2001, pp. 101 - 119.

[4] S. L. Bezrukov, J. D. Chavez, L. H. Harper, M. Röttger and U. P. Schroeder, "Embedding of hypercubes into grids", MFCS, 1998, pp. 693 - 701.

[5] S. L. Bezrukov, J. D. Chavez, L. H. Harper, M. Röttger and U. P. Schroeder, "The congestion of n-cube layout on a rectangular grid", *Discrete Mathematics*, Vol. 213, 2000, pp. 13 - 19.

[6] A. J. Boals, A. K. Gupta, and N. A. Sherwani, "Incomplete Hypercubes: Algorithms and Embeddings", The Journal of Supercomputing, Vol. 8, 1994, pp. 263 - 294.

[7] R. Caha and V. Koubek, "Optimal embeddings of generalized ladders

into hypercubes", Discrete Mathematics, Vol. 233, 2001, pp. 65 - 83.

[8] D. Chavez and R. Trapp, "The cyclic cutwidth of trees", Discrete Appl. Math., Vol. 87, 1998, pp. 25 - 32.

[9] J. A. Gallian, "A dynamic survey of graph labeling", The Electronic Journal of Coimbinatorics, Vol. 5, 2005, pp. 1.- . 20.

[10] Hsing-Lung Chen and Nian-Feng Tzeng, "A boolean expression-based approach for maximum incomplete subcube identification in faulty hypercubes", IEEE Transactions on Parallel and Distributed Systems, Vol. 8, 1997.

[11] Indra Rajasingh, J. Quadras, Paul Manuel and A. William, "Embedding of cycles into arbitrary trees", Networks, Vol. 44, 2004, pp. 173 - 178.

[12] M. Klugerman, A. Russell and R. Sundaram, "On embedding complete graphs into hypercubes", Discrete Math., Vol. 186, 1998, pp. 289 - 293.

[13] T. F. Leighton, "Introduction to parallel algorithms and architecture: Arrays, Trees, Hypercubes", *Morgan Kaufmann Publishers,* 1992, ISBN 1-55860-117-1.

[14] J.Opatrny and D. Sotteau, "Embeddings of complete binary trees into grids and extended grids with total vertex-congestion 1", Discrete Appl. Math., Vol. 98, 2000, pp. 237 - 254.

[15] Paul Manuel, Indra Rajasingh, J. Quadras and A. William, "Embedding of hypercubes into complete binary trees", Presented in the ICICS International Conference, Saudi Arabia- Submitted for publication.

[16] Paul Manuel, Indra Rajasingh, Bharati Rajan and Helda Mercy, "Exact wirelength of hypercube layout on a grid", Submited to Discrete Appl. Math.

[17] J. Xu, "Topological Structure and Analysis of Interconnection Networks", *Kluwer Academic Publishers,* 2001.